BCFW for Witten Diagrams

(a.k.a. recursion relations for correlators in AdS/CFT)

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Based on:

1. `arXiv:1011.0780` [hep-th]
2. Work in Progress

Setting

- Over the past few years, many new techniques have been developed to study scattering amplitudes in gauge and gravity theories. This has
  1. Improved our formal understanding of Quantum Field Theory.
  2. Provided efficient methods to calculate amplitudes of interest, including at the LHC.
- The most dramatic improvements — over ordinary perturbation theory — are visible in perturbative gravity.

Scattering Amplitudes in Gravity

- Perturbative gravity has an infinite number of interaction vertices with ever-increasing complexity.
- The 4-pt vertex already has 2,850 terms. So, calculating scattering amplitudes in gravity is very hard. However, the answers are very simple!
- DeWitt who first worked out some scattering amplitudes in gravity said:

  The tediousness of the algebra involved ... combined with the fact that the final results are ridiculously simple, leads one to believe that there must be an easier way.

- The BCFW recursion relations are the easier way! They relate all amplitudes to the on-shell three-point amplitude. This reduces the complexity of gravitational calculations to that of a ϕ^3 theory!

Motivation

- In this talk, I will describe how these S-matrix techniques can be generalized to quantum field theories in anti-de Sitter space.
- This is interesting from a formal perspective; these techniques were developed in flat-space and have not before been used to understand quantum field theory in curved spacetime.
- However, this also has an immediate application: the computation of correlators in AdS/CFT.

Boundary Correlators from AdS Amplitudes

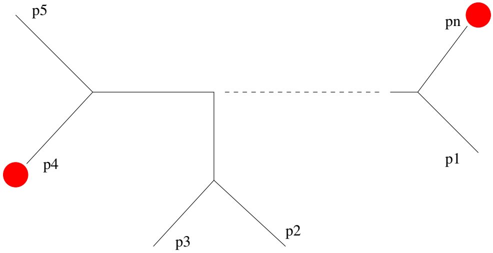
- The inordinate complexity of perturbation theory in gravity comes back to hit us, if we try and compute stress-tensor correlators.
- So, even the 4-pt function of the stress-tensor in strongly coupled $\mathcal{N} = 4$ SYM has never been computed.
- Can we use these newly developed S-matrix techniques to reduce this complexity?
- We will see that a generalization of the BCFW relations can be used to compute Witten diagrams in AdS. Just as in flat space, these formulae relate all higher order correlators to three-point functions.

Outline

Review of BCFW

- Consider a n-point graviton/gluon amplitude.

Figure: BCFW EXTENSION



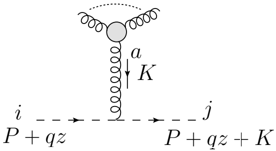
- Extend *any* two momenta on shell

$$p_4 \to p_4 + qz; \quad p_n \to p_n - qz$$
$$q^2 = q \cdot p_4 = q \cdot p_n = 0$$

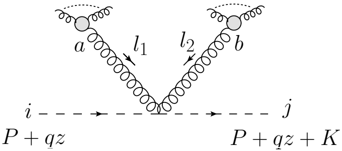
- On each side, one of two gluon-polarization vectors grows as $O(z)$. YM also has derivative interactions. Can we write down a systematic large-z expansion?

Tree Amplitudes at Large z

The physics is most transparent in q-lightcone gauge, $q \cdot A = 0$. The point is that, at large z, amplitudes are dominated by the interaction of a highly-boosted particle with a soft gas at a single point.



(a) O(z) Contribution



(b) O(1) contribution

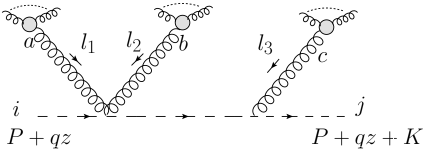


Figure: O($\frac{1}{z}$) Contribution

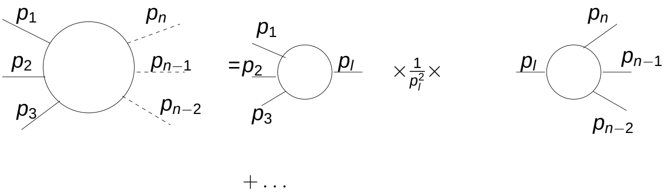
Large z Behaviour

- There are four possibilities we need to consider corresponding to two possible choices of polarization on each side.
- Naively, one might expect the amplitude to grow at large z. In fact, the amplitude goes like $O\left(\frac{1}{z}\right)$ at large z for 3 out of 4 polarizations.
- This property is very useful. The tree amplitude is a holomorphic function of z. If a holomorphic function dies off at infty, we can reconstruct it from its poles.
- Poles in the amplitude occur when an internal line goes on shell. Residues are lower pt on-shell amplitudes.
- So,

$$M(z) \sim \sum_{\text{partitions}} M_{\text{left}} \frac{1}{P_L^2(z)} M_{\text{right}}$$

Schematic BCFW

Figure: Recursion Relations



Intuition in Curved Space

- The lesson is that, in the large z limit, the amplitude is dominated by interactions at a single point.
- In this limit, we expect that the curvature of the ambient space will not be important.
- However, we do need to integrate over the various positions where this interaction can occur. (In flat space, all points are the same, so this is not necessary.)
- We will see that this intuition works out exactly.
- We find that a higher point amplitude can be broken down into the integrated product of lower point amplitudes.

Setting and Notation

- We consider AdS_{d+1} in Poincare coordinates with metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = z^{-2} \left(dz^2 + \eta_{ij} dx^i dx^j \right). \qquad (1)$$

- We use Poincare invariance in d dimensions to Fourier transform functions of x^i. We will call the conjugate variables — k_i — "momenta."

Solutions to Equations of Motion

- Recall that the eom $\Box\phi = 0$ has the following solutions, for timelike momenta ($\nu = d/2$)
  1. the non-normalizable solution $\phi = \phi_0 e^{ik\cdot x} z^\nu Y_\nu(|k|z)$, which corresponds to turning on a source on the boundary
  2. the normalizable solution $\phi = \phi_0 e^{ik\cdot x} z^\nu J_\nu(|k|z)$ which corresponds to a coherent state on the boundary.
- Yang-Mills equations in AdS are, in addition, parameterized by a transverse d dimensional vector.
- Similarly, gravity waves are parameterized by transverse traceless tensors in d-dimensions.

Propagators

- For scalars, (with $\nu = d/2$)

$$G^{\text{scal}} = \int_{k,p} \frac{e^{ik\cdot(x-x')} z^{\nu} J_\nu(pz) J_\nu(pz')(z')^{\nu}}{\left(k^2 + p^2 - i\epsilon\right)},$$

- While for gravity, we also need a projector

$$G^{\text{grav}}_{ij,kl} = \int_{k,p} \left[\frac{e^{ik\cdot(x-x')} z^{\nu-2} J_\nu(pz) J_\nu(pz')(z')^{\nu-2}}{\left(k^2 + p^2 - i\epsilon\right)} \times \frac{1}{2}\left(\mathcal{T}_{ik}\mathcal{T}_{jl} + \mathcal{T}_{il}\mathcal{T}_{jk} - \frac{2\mathcal{T}_{ij}\mathcal{T}_{kl}}{d-1}\right)\right],$$

  where $\mathcal{T}_{ij} = \eta_{ij} + k_i k_j / p^2$
- Note that at $p^2 = -k^2$, the numerator of the integrand breaks up into a sum of a product of normalizable modes.

The Objects of Study

- Consider CFT operators $O(k_{3i})$ and states s, s' that are dual, respectively, to linear combinations of normalizable modes with momenta k_{1i} and k_{2i} in the bulk. We will examine the "transition amplitude"

$$T(k_{ji})(2\pi)^d \delta^d(\sum_{ji} k_{ji}) = \langle s|O(k_{31})\dots O(k_{3n})|s'\rangle.$$

- Note that a vacuum correlator is just a special case of a transition amplitude where $|s'\rangle, \langle s|$ are both the vacuum.
- Nevertheless, it is of interest to consider these generalized objects since, even if we start out to compute vacuum correlators, the recursion relations generate these transition amplitudes.

Perturbative Prescription for Transition Amplitudes

Perturbatively, we are looking at a generalization of Witten diagrams, where some bulk-to-boundary propagators are replaced by normalizable modes.

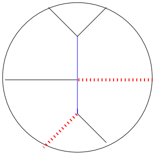


Figure: Transition Amplitude

Sketch of Derivation

- Consider a transition amplitude $T(k_1, \ldots, k_n)$.
- Extend $k_1 \to k_1 + qw, k_n \to k_n - qw$, with $q^2 = 0$ and $k_1 \cdot q = k_n \cdot q = 0$.
- The amplitude has a complicated analytic structure in w, but it can be expressed as the integral of a rational function of w.
- We can show that this integrand dies off at large w and reconstruct it using its poles at finite w.
- This allows us to derive recursion relations.

BCFW relations in AdS

- For gravitons in the bulk (corresponding to stress-tensor insertions on the boundary)

$$T(k_1, \epsilon_1 \ldots k_n, \epsilon_n) = \sum_{\{\pi\}, i, \epsilon'_i} \int \mathcal{T}^2 \frac{dp^2}{2},$$

$$\mathcal{T}^2 \equiv \frac{-iT(k_1(p), \epsilon_1, \ldots k'_i, \epsilon'_i) T(-k'_i, \epsilon'_i, \ldots k_n(p), \epsilon_n)}{(p^2 + K^2)}.$$

- Here,

$$K = k_1 + \sum_2^i k_{\pi_i}; w(p) = -(K^2 + p^2)/(2K \cdot q);$$

$$k_1(p) = k_1 + qw(p); k_n(p) = k_n - qw(p);$$

- Note that the sum runs over all ways of partitioning the momenta into two sets: $\{k_1, k_{\pi_2} \ldots k_{\pi_i}\}, \{k_{\pi_{i+1}} \ldots k_n\}$ and also over intermediate polarization vectors.

Schematic Representation

This can be seen schematically in the figure. Starting with a four-point vacuum correlator, we get the integral of the product of two three-point transition amplitudes each of which has one normalizable mode (shown by the dotted line).

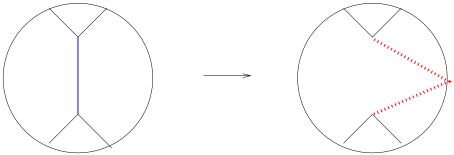


Figure: Recursion Relations

Conditions on Polarization Vectors

- The conditions on polarization vectors are stricter.
- For example for gravitons in AdS_5, we have 25 possible polarizations (5 on each side). The BCFW relations work correctly for only 6/25 possibilities.
- This is similar to what happens for massive theories in flat space.
- Fortunately, by extending different pairs of particles, we can always compute an amplitude with arbitrary polarizations.

BCFW for Susy Theories in AdS

- In flat space, the recursion relations can be extended to theories with supersymmetry by using supersymmetry to relate the scattering of matter to that of gluons or gravitons.
- AdS supergroups exist for $d \leq 6$.
- Here, we can compute correlators of operators that are in the same multiplet as a conserved current or the stress tensor.
- The trick is to parameterize the supermultiplet using smooth Grassmann parameters.

Grassmann parameterization of multiplets

- Consider the case $d = 6$ (very similar parameterizations work in $d = 4$)
- The supercharges live in a 6 dimensional chiral-spinor representation (with eigenvalues $\pm 1/2$ under Lorentz transforms in the $(2i-1, 2i)$ plane) and in an R-symmetry group $Sp(2\mathcal{N})$ where $\mathcal{N}$ is 1 or 2.
- We choose $k_1 = (1,0,0,0,0,0), k_n = (a,b,0,0,0,0), q = (0,0,0,0,1,I)$ and form the array of $4\mathcal{N}$ supercharges: $\mathcal{Q}^A_{1+} = \{Q^I_{\pm 1/2,\pm 1/2, 1/2}\}$
- Then, with $T_{--}(k_i) = T_{mn}(k_i)q^m q^n$, the components of the stress tensor multiplet can be parameterized by

$$T_i(\eta) = U_+(\eta) T_{--}(k_i) U_+(-\eta);\ U_+(\eta) \equiv e^{\mathcal{Q}^A_+ \eta_A}.$$

Calculable Correlators in Susy Theories

- So far this is very similar to Nair's "on-shell superspace" in flat space.
- However, in AdS, we can use the BCFW relations to compute any "diagonal" correlator that can be written as
$$\langle T_1(\eta) T_n(\eta) O \rangle = \langle T_{--}(k_1) T_{--}(k_n) O' \rangle,$$
- In $d = 4$, only these diagonal correlators are calculable. In $d = 6$, we can calculate a slightly larger subset that I will not describe here.
- In $d = 5$, the stress-tensor lives in a quarter-BPS multiplet, which does not allow a complete parameterization of the sort above. The set of calculable correlators is even more strongly restricted.
- So, supersymmetry does not seem as powerful as it is in flat space. (although it would be nice if there is some clever trick that we have overlooked!)

Summary

- We were able to find new recursion relations for correlation functions in conformal field theories with an AdS dual.
- These recursion relations rely on relating these to scattering amplitudes in bulk AdS.
- They relate a higher order correlator to an integrated product of two lower order correlators.
- The physical intuition is that, by extending two of the momenta in a “complex direction”, we can restrict ourselves to interactions that happen at a single point; the recursion relations are obtained by using the usual flat space recursion relations and integrating over the various points where this interaction can occur.
- Work in progress to compute concrete examples.